Combining multivariate volatility forecasts: an economic-based approach

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Abstract

We devise a novel approach to combine predictions of high dimensional conditional covariance matrices using economic criteria based on portfolio selection. The combination scheme takes into account not only the portfolio objective function but also the portfolio characteristics in order to define the mixing weights. Three important advantages are that i) it does not require a proxy for the latent conditional covariance matrix, ii) it does not require optimization of the combination weights, and iii) it holds the equally-weighted model combination as a particular case. Empirical applications involving three large data sets from different markets show that the proposed economic-based combinations of multivariate GARCH forecasts leads to mean-variance portfolios with higher risk-adjusted performance in terms of Sharpe ratio as well as to minimum variance portfolios with lower risk on an out-of-sample basis with respect to a number of benchmark specifications.

Key words: Composite likelihood, conditional correlation models, high frequency data, loss function, realized covariance.

JEL classification: C53; E43; G17.

1 Introduction

This paper is concerned with the problem of multivariate volatility forecasting. This issue permeates many important economic problems such as asset pricing, portfolio allocation and risk management and is relevant for both academics and market practitioners. We are interested in knowing if - and, more importantly, how - we can profit from combining forecasts from alternative individual models in order to improve decision making in economic problems that rely on such forecasts. Our main contribution is to define combination weights that take into account the economic decision in which forecasts will be ultimately used. We show that the proposed economically motivated approach to combine forecasts from individual models is able to outperform existing methods, and is also more flexible and easier to implement. In particular, the method does not require a proxy for the unobserved volatility process and is entirely driven by the portfolio selection problem, which is one of the most important economic problems relying on forecasts of multivariate volatilities.

Combining predictions from alternative models has been used as a forecasting approach at least since Bates and Granger (1969). In fact, the literature on forecast combinations is extensive and points to the superiority of combined forecasts with respect to single models in many different contexts (see, for example, Timmermann, 2006, and references therein). The motivation to combine forecasts comes from the idea that a linear combination of two or more forecasts may yield more accurate predictions than using only a single forecast (Granger, 1989; Newbold and Harvey, 2002; Baumeister and Kilian, 2015). Additionally, adaptive strategies for combining forecasts might also mitigate structural breaks and model
misspecification and thus lead to more accurate forecasts (Pesaran and Timmermann, 2007).

Existing evidence suggests that the combination of univariate volatility predictions leads to more accurate forecasts with respect to single models (Becker and Clements, 2008; Patton and Sheppard, 2009). The study of combined multivariate volatility predictions, however, has received much less attention. This issue is important since economic problems often require modeling and forecasting not only variances but also covariances among multiple time series. In this sense, the combination of multivariate volatility models can lead to better decision making in economic and financial problems that depend on covariance estimates such as asset pricing (Bollerslev et al., 1988; Bali and Engle, 2010), portfolio optimization (Engle and Colacito, 2006; Becker et al., 2014) and risk management (Santos et al., 2013). Engle and Colacito (2006), for instance, show that accurate covariance information will allow the investor to achieve lower volatility, higher return, or both.

Amendola and Storti (2015) pioneered the literature on combined multivariate volatility predictions extending the approach in Patton and Sheppard (2009) to a multivariate setting. Their approach estimates model combination weights via minimization of statistical loss functions, and requires high-frequency data to compute a proxy for the unobserved covariance matrix, which is specified as a realized covariance measure.

One common aspect in the existing literature on forecast combination is that it focuses on purely statistical measures of forecast accuracy to determine the relative importance of individual models in a forecast combination scheme, therefore ignoring the way forecasts affect decision-making. Nevertheless, statistical measures of forecast accuracy are not necessarily relevant to a decision maker, who is often more interested in its expected economic value (Pesaran and Skouras, 2002). While the usefulness of considering the decision-making process when evaluating forecasts is well established (Granger and Pesaran, 2000), the forecast combination literature has largely ignored this suggestion. Moreover, Patton (2011) shows that the use of imperfect volatility proxies to evaluate conditional variance forecasts can lead to undesirable outcomes, affecting the ranking of volatility forecasts. More specifically, these authors show that, for certain loss functions, an imperfect volatility forecast can be selected over the true conditional variance when noisy volatility proxies are used.

In this paper, we put forward an economic-based approach to combine multivariate volatility predictions from alternative conditional covariance models that does not require a proxy for the covariance matrix. The proposed combination rule is motivated by the fact that forecasts of covariance matrices are
often applied in portfolio selection problems, and exploits the decision-making problem in which forecasts will be used in order to determine the combination weights. More specifically, the importance of each individual model is defined by taking into account its past performance in obtaining optimal portfolios, therefore avoiding the use of imperfect covariance measures.

Another important aspect of the proposed approach is that combination weights can be calibrated in order to adjust i) the influence of the best performing models in the combination (i.e. a cross section adjustment) and ii) the importance given to the most recent observations in the calculation of portfolio performance (that is, a time series adjustment). Further intuition on the impact of these calibrations are given in Section 2.1. Finally, three major advantages of this approach are that i) it does not require a proxy for the latent conditional covariance matrix, which avoids the need to use high-frequency data ii) it does not require optimization of the combination weights, which facilitates its implementation, and iii) it holds the equally-weighted model combination as a particular case. Each of these advantages are discussed in Section 2.

The proposed approach to combine predictions of conditional covariance matrices is based on the mean-variance portfolio selection problem introduced by Markowitz (1952). Two aspects motivate our choice for adopting the mean-variance framework. First, economic applications involving multivariate volatility models often rely on the mean-variance portfolio problem; see, for example, Engle and Colacito (2006) and Engle and Sheppard (2008). Therefore, it is natural to consider this type of portfolio selection policy when devising an economic-based model combination approach for conditional covariance models. Second, the mean-variance framework is one of the milestones of modern finance theory and is widespread among academics and market participants, see Brandt (2009) for a discussion.

We consider two alternative versions of the mean-variance portfolio problem. The first is based on an investor who wishes to minimize portfolio risk subjected to a target portfolio return. In this case, the investor wishes to achieve higher risk-adjusted portfolio returns. The mean-variance problem, however, is known to be very sensitive to estimation of the mean returns (e.g. Jagannathan and Ma, 2003). Very often, the estimation error in the mean returns degrade the overall portfolio performance and introduces an undesirable level of portfolio turnover. In fact, existing evidence suggest that the performance of optimal portfolios that do not rely on estimated mean returns is better; see, for instance, DeMiguel et al. (2009). Because of that, we also consider a second type of investor who adopts the minimum variance criterion in order to decide her portfolio allocations. This portfolio policy can be seen as a particular case
of the traditional mean-variance optimization.

Our paper is related to recent studies such as Becker et al. (2014) and Amendola and Storti (2015). Although Becker et al. (2014) does not consider forecast combination, they study the ability of alternative loss functions to select the best specifications for the portfolio selection problem, and find that a likelihood-based function performs best for this purpose. However, in contrast to these approaches, we propose a covariance matrix combination strategy that does not require neither optimization of combination weights, nor a proxy for the unobserved covariance matrix, greatly facilitating its implementation in practical situations specially those that involve high dimensional problems. Moreover, instead of adopting a purely statistical criterion for estimating the mixing weights, we directly apply an economic criterion for defining, in a dynamic fashion, how much weight to place in each forecast. Therefore, our approach to combine multivariate volatility predictions is consistent with the fact that this class of models is ultimately applied to economic problems.

We test the effectiveness of the proposed forecast combination scheme using a data set composed of daily closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 over a 10-year time span. The empirical applications works as follows. First, we implement a set of 8 most popular individual multivariate GARCH specifications used in portfolio selection problems, and obtain their corresponding mean-variance and minimum variance portfolios. The multivariate GARCH specifications employed are the exponentially weighted moving average, the rolling estimator of Foster and Nelson (1996), the scalar VECH of Bollerslev et al. (1988), the orthogonal GARCH of Alexander (2001), the constant conditional correlation of Bollerslev (1990), the dynamic conditional correlation model of Engle (2002) and its asymmetric version proposed in Cappiello et al. (2006), and the dynamic equicorrelation of Engle and Kelly (2012). Second, we obtain combined multivariate volatility forecasts according to our proposed method and use these forecasts to solve the mean-variance and minimum variance portfolio problems. Third, we conduct a detailed economic evaluation of the resulting optimal portfolios and implement a test for the portfolio risk and for the portfolio risk-adjusted return measured by the Sharpe ratio (SR) based on the bootstrap procedure of Politis and Romano (1994), which allows us to formally compare optimal portfolios obtained with alternative specifications in terms of their sample characteristics. Finally, we also take into account the impact of portfolio turnover as well as the presence of transaction costs when evaluating portfolio performance.

Our results can be summarized as follows. First, when combining multivariate volatility forecasts based
on the mean-variance criterion, we observe that the resulting conditional covariance estimators yield mean-
variance portfolios with higher risk-adjusted returns measured by the SR. In particular, the out-of-sample
SR of the mean-variance portfolios (after accounting for transaction costs) obtained with the proposed
combined forecasts is 0.10, whereas the same figure for the equally-weighted combination is 0.05. Second,
when combining multivariate volatility forecasts based on the minimum variance criterion, we observe that
the minimum variance portfolios obtained with the combined estimators are statistically less risky with
respect to all individual models considered. Specifically, we find that the proposed combination delivered
minimum variance portfolios with the lowest level of risk among all other specifications (0.42%), whereas
the same figure for the equally-weighted combination is 0.44% and for the best performing individual
model is 0.45%.

Additionally, we are also interested in evaluating the extent to which our proposed method to combine
multivariate volatility forecasts outperforms existing methods when applied to the problem of portfolio
selection. For that purpose, we implement the method proposed in Amendola and Storti (2015), which
consists in estimating model combination weights via minimization of alternative loss functions considered
in Laurent et al. (2012, 2013). Note that this method requires a proxy for the unobserved realized
covariance matrix. We follow Amendola and Storti (2015) and compute realized covariance measures
based on one-minute intraday data. Results suggest that, despite the larger computational burden to
implement this method, the use of economic criteria to devise a forecast combination scheme is more
appropriate when the ultimate goal is to use the resulting combined forecasts to solve an economic
problem.

Finally, we conduct a robustness check to investigate the comparative performance of the proposed
forecast combinations by considering two alternative data sets from the international equities markets.
The first alternative data set consists of daily closing prices of 45 stocks belonging to the European stock
market index Eurostoxx whereas the second alternative data set consists of daily closing prices of 17 stocks
belonging to the Asian stock market index STI. The results obtained with these two data sets are reassuring
and points to the superiority of the mean-variance and minimum variance portfolios obtained with the
proposed forecast combinations with respect to those obtained with the equally-weighted combination as
well as with respect to the vast majority of individual models.

The rest of the paper is organized as follows. In section 2 we detail the proposed forecast combination
rule and revise the existing methods. Section 3 provides an empirical application. Finally, section 4
concludes.

2 Combining multivariate volatility forecasts

In this section, we detail the proposed approach to combine multivariate volatility predictions in a dynamic fashion. To solve a variety of important problems in finance, it is common to choose a conditional covariance specification in order to account for the time variation in second order moments; see, for instance, Engle and Colacito (2006) and Becker et al. (2014). In most practical situations, however, the volatility process is subject to changes and the investor faces uncertainty on which is the most appropriate specification to model and to forecast covariances at a given point in time. For instance, should the investor consider time varying or time invariant conditional correlations? Are more parameterized specifications better than more parsimonious ones? How to handle this uncertainty? All these questions are typically answered in the literature by horse-racing many single models; see, for instance, Engle and Colacito (2006) and Engle and Sheppard (2008). We, on the other hand, consider the possibility of improving portfolio performance via model-based forecast combinations. More specifically, we consider the case in which there exists $M$ alternative candidates. In this case, the combined conditional covariance estimator, $H_t^{Comb}$, is defined as

$$H_t^{Comb} = \lambda_{1,t} H_1^t + \ldots + \lambda_{M,t} H_M^t,$$

where $H_m^t$ denotes conditional covariance matrix of the $m$-th candidate model and $\lambda_{m,t}$ is the corresponding combination weight. Except when otherwise noted, we focus on the case in which $\sum_{m=1}^M \lambda_{m,t} = 1$ (i.e. convex combinations), and $\lambda_{m,t} \geq 0 \forall m$ (non negative combination weights). It is worth noting that imposing non negative weights in (1) guarantees that the resulting $H_t^{Comb}$ is positive-definite provided that individual models also deliver positive-definite conditional covariance matrices.

The most important aspect when combining alternative individual forecasts is to specify a combination vector $\lambda_t = \{\lambda_{1,t}, \ldots, \lambda_{M,t}\}$ in (1), that is, to decide how much weight to place in each forecast. The literature on forecast combinations offers alternative options to determine $\lambda_t$. Next we detail our approach based on portfolio selection problems, and also review the existing approach to combine multivariate volatility forecasts.
2.1 Economic-based forecast combinations

Differently from the existing literature, we propose to specify the combination weights $\lambda_m$ in (1) so that it incorporates the economic decision in which the forecast of the covariance matrix will be used. More specifically, $\lambda_m$ will be determined based on the performance of each individual model in mean-variance and minimum variance portfolio optimization problems. Using these economic criteria has at least four appealing features. First, it is consistent with the fact that a multivariate volatility forecasts is not an end in itself, and are ultimately applied to economic problems, see, for example, Granger and Machina (2006) and Elliott and Timmermann (2008). Second, it is based on well established portfolio selection problems with widespread interest among academics and market participants, see, for instance, Brandt (2009) and references therein. Third, these economic optimization problems posses closed-form solutions, thus avoiding numerical optimization of loss functions in order to obtain the combination weights as in Amendola and Storti (2015). Fourth, the combination scheme proposed does not need a proxy for the unobserved covariance matrix, bypassing the need to use high-frequency data to estimate the latent volatility as in Patton and Sheppard (2009) and Amendola and Storti (2015). Next we detail two different economic gain functions used to construct the combined conditional covariance estimator in (1).

2.1.1 Mean-variance forecast combination

We consider an investor who allocates her wealth in $N$ alternative risky assets. In order to choose the weights $w_i$ for $i = 1, \ldots, N$ of each asset in the portfolio, we assume the investor adopts the mean-variance portfolio policy. In this setting, the investor wishes to minimize portfolio risk subjected to a target portfolio return. This portfolio optimization problem is defined as

$$\min_{w \in \mathbb{R}^N} w_i' H_t^m w_t$$

subject to

$$w_i' \mu = \mu_0$$
$$\sum_{i=1}^N w_i = 1,$$

where $w_t$ is the vector of portfolio weights for time $t$ chosen at time $t-1$ and $H_t^m$ is a positive-definite conditional covariance matrix of asset returns obtained with the $m$-th candidate model for time $t$ and forecasted at time $t-1$. $\mu$ is the vector of expected returns and $\mu_0$ is the required return. In our empirical
implementation, we define $\mu_0$ as the return of the equally-weighted portfolio. The solution to (2) is given by

\[ w_t = \left( H_m^{t-1} \right)^{-1} \left( \mu - B \right), \]

where $A = \mu \left( H_m^{t-1} \right)^{-1}$, $B = \mu \left( H_m^{t-1} \right)^{-1} \iota$, and $\iota$ is a vector of ones; see Cochrane (2009).

Upon solving the portfolio problem in (2) for a sample of $T$ observations and adopting a given specification for the conditional covariance matrix $H_m^{t-1}$, the investor is interested in computing the portfolio risk-adjusted return, which is usually measured by the SR and is defined as

\[ SR_m = \frac{\hat{\mu}_m}{\hat{\sigma}^2_m}, \]

where $\hat{\mu}_m = \frac{1}{T} \sum_{t=1}^T w_t' R_{t+1}$ is the average portfolio return and $\hat{\sigma}^2_m = \frac{1}{T-1} \sum_{t=1}^T (w_t' R_{t+1} - \hat{\mu}_m)^2$ is the portfolio variance, where $R_{t+1}$ is the vector of asset returns at time $t + 1$. Instead of computing both $\hat{\mu}_m$ and $\hat{\sigma}^2_m$ as the standard sample mean and sample variance, we follow Stock and Watson (2004) and Genre et al. (2013) and consider a more general expression for these two performance metrics by introducing a discount (or forgetting) parameter $\delta$, i.e.

\[
\begin{align*}
\hat{\mu}_m &= \frac{1}{T-1} \sum_{t=1}^T \delta^{T-1-t} \left( w_t' R_{t+1} \right), \\
\hat{\sigma}^2_m &= \frac{1}{T-1} \sum_{t=1}^T \delta^{T-1-t} (w_t' R_{t+1} - \hat{\mu}_m)^2.
\end{align*}
\]

Values of $\delta$ which are below unity assign lower weights to older data in the calculation of portfolio average returns and portfolio variance. The case where $\delta = 1$ corresponds to no discounting and is equivalent to the common expression of the sample mean and sample variance. Diebold and Pauly (1987) argues that values of $\delta$ lower than 1 (therefore decreasing the importance of older observations), can be useful to account for stylized facts in financial time series, such as time-varying conditional heteroskedasticity and structural breaks; see Andreou and Ghysels (2009).

In order to gain intuition on the effects of changing the values of the $\delta$ in (3), we plot in Figure 1 the values of the discount factor, $\delta^t$, for a sample with $t = 1, \ldots, 100$ observations, where the $t = 1$ is the most recent observation. Figure 1 plots the resulting discount factor when $\delta = \{1, 0.95, 0.90, 0.85\}$. As expected, we observe an exponential decay in the importance given to the most recent observations when the value of $\delta$ decreases, implying that the smaller $\delta$ is, the faster old observations are “forgotten”. Specifically, the discount factor $\delta^t$ implies that the observation $t$ has weight $\delta^t$ in the estimation of portfolio average returns and portfolio variance in (3).

In a mean-variance context, the combined conditional covariance estimator in (1) should emphasize the individual models that yield mean-variance portfolios with higher risk-adjusted returns (i.e. higher
SR and penalize those that yield portfolio with lower risk-adjusted returns. In this case, upon solving (2) using each of the candidate models $H_t^m$ for $m = 1, \ldots, M$, the performance-based model combination vector $\lambda_m$ in (1) can be defined as

$$
\lambda_m = \left( \frac{\hat{\mu}_m / \hat{\sigma}_m}{\sum_{m=1}^M \left( \frac{\hat{\mu}_m / \hat{\sigma}_m}{\hat{\sigma}_m} \right)} \right)^{\alpha},
$$

where $\hat{\mu}_m$ and $\hat{\sigma}_m^2$ are, respectively, the $\delta$-adjusted portfolio average return and portfolio variance defined in (3). In order to rule out the possibility of selecting models that generate portfolios with negative SR, one can set $\lambda_m = 0$ if $SR_m < 0$. We refer to this approach as the mean-var($\delta$) combination rule.

The inspiration for defining the economic-based model combination in (4) comes from Kirby and Ostdiek (2012), who devised a portfolio policy that takes into account the inverse of the sample variance of the individual assets as well as their risk-adjusted performance in order to define how much weight to put in each individual asset.

The mean-var($\delta$) strategy in (4) and (3) belongs to a more general class of mean-var combination with weights of the form

$$
\lambda_m = \left( \frac{\hat{\mu}_m / \hat{\sigma}_m}{\sum_{m=1}^M \left( \frac{\hat{\mu}_m / \hat{\sigma}_m}{\hat{\sigma}_m} \right)} \right)^{\eta},
$$

The idea behind this generalization is straightforward. The tuning parameter $\eta \geq 0$ determines how aggressively we adjust the mixing weights in response to changes in the realized portfolio SR obtained with each of the candidate models. As $\eta \to 0$ we recover the equally-weighted model combination, and as $\eta \to \infty$ the weight on the model that yields the highest realized SR approaches 1. Thus, large values of $\eta$, can shrink the combination weights towards the best performing models. For example, Granger and Jeon (2004) argue that poor performing models can substantially worsen forecasting performance. The additional shrinkage implied via $\eta$ could provide some forecast benefits via reducing the importance of the worst performing models.

In order to gain intuition on the effects of changing the values of the $\eta$ in the forecast combination weights in (5), we plot in Figure 2 the results of a small simulated experiment that works as follows. Say there are $m = 1, \ldots, 10$ alternative individual candidate models that delivered mean-variance portfolios according to (2) with SR ranging from 0.1 to 1.0, with increments of 0.1, such that $SR_{m1} = 0.1, \ldots, SR_{m10} = 1.0$. We then compute the resulting combination weights using (5) for $\eta = \{0, 2, 4, 6, 8, 10\}$. As expected, Figure 2 shows that when $\eta = 0$ the combination rule in (5) places
equal weights to all models, regardless of their performance. However, when the value of $\eta$ increases, the weights on the worst performing models (the ones with lower SR) decrease exponentially, evidencing how $\eta$ can be used as a means of trimming forecasts.

We refer to the mixing in (5) as the mean-var($\delta, \eta$) combination rule. Finally, substituting equation (5) into (1) yields a more general expression for the mean-var($\delta, \eta$) combined estimator:

$$H_{t}^{\text{Comb}} = \frac{1}{\sum_{m=1}^{M} \left( \frac{\hat{\mu}_{m}}{\hat{\sigma}_{m}} \right)^{\eta}} \sum_{m=1}^{M} \left( \frac{\hat{\mu}_{m}}{\hat{\sigma}_{m}} \right)^{\eta} H_{t}^{m}. \quad (6)$$

### 2.1.2 Minimum variance forecast combination

We now consider an alternative performance-based model combination rule based on the minimum variance portfolio policy. A very large body of literature in portfolio optimization considers this particular policy; see, for instance, Clarke et al. (2006, 2011) for extensive practitioner-oriented studies on the performance and on the composition of minimum variance portfolios. This policy can be seen as a particular case of the traditional mean-variance optimization. The mean-variance problem, however, is known to be very sensitive to estimation of the mean returns (e.g. Jagannathan and Ma, 2003). Very often, the estimation error in the mean returns degrade the overall portfolio performance and introduces an undesirable level of portfolio turnover. In fact, existing evidence suggest that the performance of optimal portfolios that do not rely on estimated mean returns is better; see, for instance, DeMiguel et al. (2009).

The minimum variance portfolio problem is defined as

$$\min_{w \in \mathbb{R}^{N}} w' H_{t}^{m} w_{t}$$

subject to

$$\sum_{i=1}^{N} w_{i} = 1,$$

where $w_{t}$ is the vector of portfolio weights for time $t$ chosen at time $t - 1$ and $H_{t}^{m}$ is a positive-definite conditional covariance matrix of asset returns obtained with the $m$-th candidate model for time $t$ and forecasted at time $t - 1$. The solution to (7) is defined as $w_{t} = \nu'(H_{t}^{m})^{-1} / \nu'(H_{t}^{m})^{-1} \nu$, where $\nu$ is a vector of ones.

Upon solving the portfolio problem in (7) for a sample of $T$ observations and adopting a given specifi-
cation for the conditional covariance matrix $H^m_t$, the investor computes the $\delta$-adjusted portfolio variance as defined in (3). In the minimum variance context, the combined conditional covariance estimator in (1) should reward the models that yield minimum variance portfolios with lower risk (i.e. lower portfolio variance) and penalize those that yield portfolios with higher risk. In this case, upon solving (7) using each of the model candidates $H^m_t$ for $m = 1, \ldots, M$ and accounting for the time series adjustment by means of the $\delta$ parameter as well as for the cross section adjustment by means of the $\eta$ parameter, the economic-based minimum variance model combination vector $\lambda_m$ in (1) can be defined as

$$
\lambda_m = \frac{\left(\frac{1}{\hat{\sigma}^2_m}\right)^\eta}{\sum_{m=1}^M \left(\frac{1}{\hat{\sigma}^2_m}\right)^\eta}, \quad m = 1, \ldots, M.
$$

(8)

We refer to the mixing in (8) as the min-var($\delta, \eta$) combination rule. Finally, substituting equation (8) into (1) yields a more general expression for the min-var($\delta, \eta$) combined estimator:

$$
H^{Comb}_t = \frac{1}{\sum_{m=1}^M \left(\frac{1}{\hat{\sigma}^2_m}\right)^\eta} \sum_{m=1}^M \left[\left(\frac{1}{\hat{\sigma}^2_m}\right)^\eta H^m_t \right].
$$

(9)

2.1.3 Important remarks

The combined conditional covariance estimators in (6) and in (9) use the past performance of each individual model in delivering portfolios with higher risk-adjusted performance and lower portfolio risk, respectively, in order to define the combination weights. The aggressiveness in the allocation across alternative models is calibrated by the tuning parameter $\eta$. Moreover, the importance given to the most recent observations in the calculation of portfolio variances is calibrated by the discount factor $\delta$. Therefore, while the $\eta$ parameter performs a cross section adjustment on the aggressiveness in the allocation across alternative models, the discount factor $\delta$ performs a time series adjustment as it controls for the importance of the most recent observations.

It is also worth emphasizing three important aspects of the model combination in (6) and (9). First, the mixing does not require a proxy for the latent conditional covariance matrix. This proxy is usually defined by means of a realized covariance matrix based on high-frequency data as in Laurent et al. (2012), Laurent et al. (2013), and Becker et al. (2014). The proposed approach, on the other hand, dispenses the use of intraday data, and can be implemented with data sampled at any frequency. Second, the mixing
rule does not require optimization of the combination weights, which facilitates its implementation in practice.

Finally, the third important aspect of the proposed approach to combine multivariate volatility predictions is that it holds the equally-weighted model combination as a particular case (when \( \eta = 0 \)). This specific model combination is found to outperform more sophisticated combination schemes in many contexts; see, for instance, Clemen (1989), De Menezes et al. (2000), Wallis (2011), and Genre et al. (2013). Some authors have argued this result is due to the instability of combination weights, which can deteriorate the performance of optimal combinations; see Kang (1986). This instability has its roots in the sampling error, which contaminates the estimated weights and is exacerbated by the collinearity that typically exists among primary forecasts (Diebold, 1989). The imposition of equal weights eliminates variation in the estimated weights and increases robustness with respect to model uncertainty, time-variation of parameters, and estimation errors that arise when combination weights have to be estimated (Palm and Zellner, 1992). Finally, Armstrong (2001) and Timmermann (2006) establish conditions under which adopting the equally-weighted model combination is optimal. Timmermann (2006), for instance, argues that equal weights are optimal in situations with an arbitrary number of forecasts when the individual forecast errors have the same variance and identical pairwise correlations. These arguments motivate us to choose the equally-weighted model combination (when \( \eta = 0 \)) as the main benchmark among all individual and combined covariance specifications considered in the paper.

2.2 Review of existing approaches

The problem of combining univariate volatility forecasts has been initially considered in Becker and Clements (2008) and Patton and Sheppard (2009). The approach developed in Patton and Sheppard (2009) consists in estimating an optimal combination vector by minimizing a loss function with respect to a proxy for the true unobserved volatility, which is specified as a realized volatility measure based on intraday data. This approach has been generalized to the multivariate case in Amendola and Storti (2015), which is based on the minimization of multivariate loss functions considered in Laurent et al. (2012, 2013) and also requires a proxy for the true unobserved covariance matrix. Specifically, the estimated vector of combination weights is given by

\[
\hat{\lambda}_t = \arg \min_{\lambda} \sum_{t=1}^{T} L \left( \Sigma_t, H_t^1, \ldots, H_t^M; \lambda \right),
\]
where \( L(\cdot) \) is an appropriately chosen loss function and \( \Sigma_t \) is the multivariate volatility proxy. Similar as in Laurent et al. (2012, 2013), Becker et al. (2014), and Amendola and Storti (2015), we define the multivariate volatility proxy using a realized covariance estimator based on intraday data. For that purpose, we follow de Pooter et al. (2008) and Liu (2009) and consider a class of realized covariance estimators that includes not only intraday data but also the overnight volatility. In this case, the daily realized covariance matrix \( \Sigma_t \) is defined as

\[
\Sigma_t = r_{t,c-o} r'_{t,c-o} + \sum_{i=2}^{I} r_{t-1+i h,h} r'_{t-1+i h,h},
\]

where \( r_{t,c-o} \) is the \((N \times 1)\) vector of close-to-open (overnight) returns from day \( t-1 \) (close) to day \( t \) (open). Martens (2002) and Hansen and Lunde (2005) point out that the inclusion of the overnight volatility is important because during the close-to-open hours, no data are recorded but the price of the asset will still be responding to news.\(^4\)

We follow the same approach of Amendola and Storti (2015) and obtain realized covariances according to (11) for a sampling frequency of \( h = 1 \) minute such that the corresponding \( I \) intraday intervals completely cover the trading day.\(^5\) Under suitable conditions, \( \Sigma_t \) converges to the true unobservable covariance matrix as the the sampling frequency of intraday returns increase \( (h \to 0) \); see Barndorff-Nielsen and Shephard (2004). Amendola and Storti (2015) point out that when estimating the combination vector according to (10), removing the restriction \( \sum_{m=1}^{M} \lambda_{m,t} = 1 \) helps correcting for the presence of bias in the individual candidate forecasts. In our empirical implementation discussed in Section 3 we follow this suggestion and remove this restriction.

We consider a set of alternative loss functions \( L(\cdot) \) in order to estimate the combination weights used to construct the combined alternative multivariate volatility predictions according to (1). The choice of the loss function are inspired in Laurent et al. (2012, 2013), Becker et al. (2014) and in Amendola and Storti (2015). In particular, we estimate the combination weights considering the following loss functions:

**Euclidean distance**

\[
L_E = \text{vech} (\Sigma_t - H_t)' \text{vech} (\Sigma_t - H_t)
\]

**Frobenius distance**

\[
L_F = \text{Tr} \left[ (\Sigma_t - H_t)' (\Sigma_t - H_t) \right]
\]
Stein loss

\[ L_S = \text{Tr} \left[ H_t^{-1} \Sigma_t \right] - \log |H_t^{-1} \Sigma_t| - N \]

Asymmetric loss

\[ L_{\text{Asym}} = \frac{1}{b(b-1)} \text{Tr} \left[ \Sigma_t^b - H_t^b \right] - \frac{1}{(b-1)} \text{Tr} \left[ H_t^{b-1} (\Sigma_t - H_t) \right], \quad b \geq 3. \]

We refer the reader to Laurent et al. (2012, 2013) for detailed discussion on the properties of these loss functions. We follow Laurent et al. (2012) and set \( b = 3 \) when implementing the asymmetric loss function. Moreover, all loss functions are normalized to zero when there is perfect fit.

Despite the fact that the realized covariance matrix \( \Sigma_t \) in (11) is only an estimate of the true volatility, high frequency data to compute this measure is not always available, which creates difficulties to the implementation of this method. Furthermore, the resulting combination weights are based on general loss functions and do not incorporate any information about the decision making task in which forecasts will be ultimately used. Therefore, there is no guarantee that the combined forecasts will perform well when applied to an economic problem.

2.3 Individual candidate models

In order to implement the proposed combination rules defined in (6) and (9), as well as the model combination approach discussed in Section 2.2, it is necessary to implement a set of individual candidate models. For that purpose, we assume that the multivariate system of asset returns is conditionally heteroskedastic and follows \( R_t = z_t (H_t^m)^{1/2} \). To model \( H_t^m \), it is possible to consider a broad set of conditional covariance specifications including multivariate GARCH and SV models; see Asai et al. (2006), Bauwens et al. (2006), Silvennoinen and Teräsvirta (2009), and Chib et al. (2009). In this paper, we follow Engle and Sheppard (2008) and Becker et al. (2014) and restrict our attention to the former class and implement a set of \( M = 8 \) alternative multivariate GARCH specifications. Two aspects motivate our choice for these specifications. First, they are widely used in portfolio selection problems. Second, they are feasible to be implemented in high dimensional problems such as the ones considered in Section 3. Next we detail each of the 8 specifications considered in the paper:
Exponentially weighted moving average (EWMA): The EWMA model is defined as

$$H_t = \alpha R_{t-1}'R_{t-1} + (1 - \alpha) H_{t-1},$$

where $\alpha$ is a nonnegative parameter. When the $\alpha$ is set to a fixed value of 0.04, the EWMA is equivalent to the popular Riskmetrics approach. Zaffaroni (2008) shows that although it permits sizable computational gains and provide a simple way to impose positive semi-definitiveness of the resulting conditional covariance matrices, the Riskmetrics delivers non-consistent estimates. Therefore, in our implementation of the EWMA specification the parameter $\alpha$ is estimated via maximum likelihood; see details below.

Optimal rolling estimator (ORE): The general rolling estimator is defined as $H_t = \sum_{k=1}^{\infty} \Omega_{t-k} \odot R_{t-k}'R_{t-k}$, where $\Omega_{t-k}$ is a symmetric matrix of weights and $\odot$ denotes the element-by-element multiplication. This structure admits a wide range of potential weighting schemes. Foster and Nelson (1996) show that the optimal strategy is to let the weights decline in an exponential fashion as the magnitude of $k$ increases. Their procedure, however, implies a different decay rate for each element of the conditional covariance matrix $H_t$. Because this makes it difficult to ensure that the resulting matrix is positive definite, we follow Fleming et al. (2001, 2003) and impose the restriction $\Omega_{i,j,t} = \Omega_t$ for all $i$ and $j$. In this case, the optimal weighting scheme is given by $\Omega_{t-k} = \alpha \exp (-\alpha k)' \iota$ where $\alpha$ is the decay rate and $\iota$ is a vector of ones. Therefore, the rolling estimator can be rewritten as

$$H_t = \alpha \exp (-\alpha) R_{t-1}'R_{t-1} + \exp (-\alpha) H_{t-1},$$

(12)

where $\alpha$ is a nonnegative parameter which is estimated via maximum likelihood; see details below. In (12) a single parameter ($\alpha$) controls the rate at which the weights decay with the lag length. This parsimony facilitates estimation specially when the dimension is high. The ORE specification has been applied in many portfolio selection problems such as in Fleming et al. (2001, 2003) and Pooter et al. (2008). Fleming et al. (2003), in particular, point out that covariance matrix forecasts based on the ORE specification results in better portfolios in comparison to those obtained with other (unrestricted) multivariate GARCH models. The authors argue that the smoothness of the rolling estimator as the main reason for this.
Scalar VECH: The scalar VECH specification of Bollerslev et al. (1988) is defined as

\[ H_t = C'C + \alpha R_{t-1}R_{t-1}' + \beta H_{t-1}. \]

Instead of estimating \( N(N + 1)/2 \) unique elements of \( C \), we employ the variance targeting technique as suggested in Engle and Mezrich (1996). The general idea is to estimate the intercept matrix by an auxiliary estimator that is given by \( \hat{C}'\hat{C} = \bar{S}(1 - \alpha - \beta) \), where \( \bar{S} = \frac{1}{T} \sum_{t=1}^{T} R_t R_t' \), thus yielding the variance-targeting scalar VECH model

\[ H_t = \bar{S}(1 - \alpha - \beta) + \alpha R_{t-1}R_{t-1}' + \beta H_{t-1}, \quad (13) \]

which is covariance-stationary provided that \( \alpha + \beta < 1 \).

Orthogonal GARCH (O-GARCH): The O-GARCH model of Alexander (2001) belongs to a class of factor models and is able to achieve significant computational gains via dimensionality reduction. The O-GARCH model is given by \( \Sigma_t = W\Omega_t W \), where \( W \) is a \( N \times k \) matrix whose columns are given by the first \( k \) eigenvectors of the \( t \times N \) matrix of asset returns, and \( \Omega_t \) is a \( k \times k \) diagonal matrix whose elements are given by \( h_{fkt} \) where \( h_{fkt} \) is the conditional variance of the \( k \)-th principal component and follows a GARCH(1,1) process. We follow Engle and Sheppard (2008) and implement the O-GARCH model using 3 principal components.

Conditional correlation models: This class of models is defined as \( H_t = D_t\Psi_t D_t \), where \( D_t \) is a \( N \times N \) diagonal matrix with diagonal elements given by \( h_{i,t} \), where \( h_{i,t} \) is the conditional variance of the \( i \)-th asset and follows a GARCH(1,1) process, and \( \Psi_t \) is a symmetric conditional correlation matrix with elements \( \rho_{ij,t} \), where \( \rho_{ii,t} = 1 \), \( i, j = 1, \ldots, N \). We consider 4 alternative specifications to model \( \Psi_t \): (i) the constant conditional correlation (CCC) model of Bollerslev (1990), (ii) the dynamic conditional correlation (DCC) model of Engle (2002), (iii) the asymmetric DCC (ASYDCC) of Cappiello et al. (2006), (iv) the dynamic equicorrelation (DECO) model of Engle and Kelly (2012). Engle and Colacito (2006) and Engle and Sheppard (2008) study the performance of alternative conditional correlation models in portfolio selection problems.

Multivariate GARCH models are typically estimated via quasi maximum likelihood (QML). However,
this estimator is found to be severely biased in large dimensions; see, for instance, Engle et al. (2008) and Hafner and Reznikova (2012). In this paper, the parameters of the EWMA, ORE, and VECH specifications are estimated with the composite likelihood (CL) method proposed by Engle et al. (2008). As for the conditional correlation models, their estimation can be conveniently divided into volatility part and correlation part. The volatility part refers to estimating the univariate conditional variances which is done by QML assuming Gaussian innovations. The parameters of the correlation matrix in the DCC and ASYDCC models are estimated using the CL method. As pointed out by Engle et al. (2008), the CL estimator provides more accurate parameter estimates in comparison to the two-step procedure proposed by Engle (2002), especially in large problems.

Finally, it is important to emphasize that other multivariate GARCH specifications, apart from those considered in this paper, have been proposed in the literature; see Bauwens et al. (2006) and Silvennoinen and Teräsvirta (2009) for reviews. In this sense, the set of individual model candidates considered in this paper can be increased in many alternative ways. Important contributions in the literature on multivariate volatility models include the GO-GARCH model of Van der Weide (2002), the regime switching DCC of Pelletier (2006), the Wishart stochastic volatility model of Philipov and Glickman (2006a,b), the semiparametric model for the correlation dynamics of Hafner et al. (2006), the generalized DCC model of Billio and Caporin (2009) and Hafner and Franses (2009), the factor-DCC model of Zhang and Chan (2009), the DCC-MIDAS of Colacito et al. (2011), the HEAVY models of Noureldin et al. (2012), the factor-spline-GARCH model of Rangel and Engle (2012), the heterogeneous ASYDCC model of Asai (2013), the multivariate rotated ARCH model of Noureldin et al. (2014), the realized beta GARCH model of Hansen et al. (2014), the dynamic factor multivariate GARCH model of Santos and Moura (2014), the smooth transition conditional correlation model of Silvennoinen and Teräsvirta (2015), the COMFORT models of Paolella and Polak (2015), among many others; see also Almeida et al. (2015) for a recent discussion on the tradeoff between feasibility and flexibility in multivariate GARCH models. Even though we consider these alternative specifications to be very interesting, we prefer to limit the computational effort of performing multiple expanding window estimations by focusing on a model set that includes 8 of the most widely used specifications.
3 Empirical application

In this Section, we discuss the empirical application carried out in the paper. The empirical exercise aims at investigating the performance of both mean-variance and minimum variance portfolios obtained with the individual models discussed in Section 2.3 as well as with the existing forecast combination approach discussed in Section 2.2 and the proposed mean-var(δ, η) and min-var(δ, η) approaches discussed in Section 2.1. For that purpose, we first present the data, the implementation details and the methodology to evaluate portfolio performance. Finally, we discuss the results.

3.1 Data

The data set consists of closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 from 01/01/2004 until 31/12/2013. The data set has \( T = 2510 \) observations. In order to implement the existing methods to combine multivariate volatility forecasts proposed in Amendola and Storti (2015), we follow these authors and use 1-minute intraday data from 9.30 a.m. to 4.00 p.m. to construct the multivariate volatility proxy according to (11). The data provider is TickDataMarket.com. Details regarding the construction of the volatility proxy are provided in Section 2.2.

3.2 Implementation details

In order to implement the proposed mean-var(δ, η) and min-var(δ, η) forecast combinations discussed in Section 2.1, we employ a similar approach as in Kirby and Ostdiek (2012) and Genre et al. (2013) and consider two alternative set of values for the \( \delta \) and \( \eta \) parameters. In particular, we first consider the baseline case in which \( \delta = 1 \) (i.e. no discounting) and \( \eta = 1 \). Note that using \( \delta = 1 \) in (3) implicitly assumes that the expected portfolio return and its variance are fixed but unknown quantities to be estimated from \( T - 1 \) sample observations. In this case, changes in \( \hat{\mu}_m \) and \( \hat{\sigma}^2_m \) as \( T \to \infty \) are not structural but rather represent the convergence in probability of \( \hat{\mu}_m \) and \( \hat{\sigma}^2_m \) to \( \mu_m \) and \( \sigma^2_m \), respectively. Diebold and Pauly (1987) argue that using \( \delta = 1 \) when the true but unknown quantities are not fixed over time is severely suboptimal, and show that using \( \delta < 1 \) can significantly reduce the forecast error variance when structural changes occur. Given the evidence of structural breaks and time-varying conditional heteroskedasticity in financial time series (see, among others, Andreou and Ghysels, 2009), it is interesting to consider \( \delta < 1 \). Moreover, several authors find that including very poor models in an equal-weighted combination can substantially worsen forecasting performance of the combination and recommend eliminating models.
with weights close to zero as a means of reducing parameter estimation errors (see, for example, Granger and Jeon, 2004; Timmermann, 2006). Thus, shrinking the weights towards best performing models using \( \eta > 1 \) might provide additional forecast benefits.

In order to reduce the impact of structural breaks when estimating \( \hat{\mu}_m \) and \( \hat{\sigma}^2_m \), and to place less weight on the worst performing models, we also consider \( \delta = 0.90 \) and \( \eta = 2 \). Selecting \( \delta = 0.90 \) means that observations of the past month are the most influential ones. More specifically, when \( \delta = 0.90 \) the most recent observation (\( t = 1 \)) will have weight 0.90 whereas observations corresponding to the past one and two months (\( t = 21 \) and \( t = 42 \), respectively) will have weights 0.11 and 0.01, respectively, in the calculation of portfolio average returns and portfolio variances according to (3). This implies that the discount factor die out quickly, which is useful when dealing with unstable financial data.

To motivate the choice of \( \eta = 2 \), it is useful to return to the example provided in Section 2.1.1 and illustrated in Figure 2. In that example we see that when \( \eta = 0 \) all models have the same weight (10%), and when \( \eta = 1 \) all models have weights proportional to their recent performance. This means that when \( \eta = 1 \), the weight of the worst model is 1.8% and that of the best performing model is 18%. However, when \( \eta = 2 \) the worst and best models have, respectively, weights of 0% and 26%, and the top five models receive 85% of the total allocation. Therefore, \( \eta = 2 \) not only shrinks the allocation across alternative models towards the best performing models, but also allow elimination of the worst performing specifications. Obviously, increasing \( \eta \) further would penalize even more the worst models such that when \( \eta \to \infty \), the forecast combination rule becomes a model selection procedure. However, as argued by Koop and Korobilis (2012), \( \eta \to \infty \) can result in a continuous change of the forecasting model. In a portfolio allocation strategy, this frequent model switch can generate large changes in portfolio weights, and thus increases in transactions costs. Therefore, selecting \( \eta = 2 \) seems a good compromise between trimming of the worst models and portfolio weight variability.

In any case, to check the robustness of the choice of these specific values for \( \delta \) and \( \eta \), we also provide in Section 3.4.1 a detailed sensitivity analysis by considering the performance of mean-var(\( \delta, \eta \)) and min-var(\( \delta, \eta \)) combinations based on alternative values of \( \delta \) and \( \eta \). Finally, we also implement the forecast combination method proposed in Amendola and Storti (2015) and presented in Section 2.2 for the following loss functions: Euclidean (\( L_E \)), Frobenius (\( L_F \)), Stein (\( L_S \)) and asymmetric (\( L_{Asym} \)).

To obtain one-step-ahead forecasts of the individual models and to obtain the combined forecasts, we employ an expanding window procedure that works as follows. Departing from the first \( t = 1500 \)
observations, all models described in section 2.3 are estimated and their corresponding one-step-ahead forecasts of the conditional covariance matrix are obtained. We obtain mean-variance and minimum variance portfolio weights for each of the models along the \( t \) observations and compute the one-step-ahead forecast of the combined conditional covariance matrix using the mean-var\((\delta, \eta)\) and min-var\((\delta, \eta)\) combination rule in (6) and (9). Moreover, we obtain the mean-variance portfolios for each mean-var\((\delta, \eta)\) forecast combination and obtain minimum variance portfolios for each min-var\((\delta, \eta)\) forecast combination. Finally, we add one observation to the estimation window and repeat the process until the end of the data set is reached. We end up with a sample of \( T - t = 1010 \) out-of-sample forecasts. A similar procedure is used to implement the existing approaches to combine multivariate volatility forecasts discussed in Section 2.2.

It is important to notice that the implementation procedure described above implies that the each candidate model is evaluated in terms of its in-sample portfolio performance. This choice is motivated by two reasons. First, it simplifies implementation as the same estimation window used to estimate the parameters of each individual model is also used to evaluate the portfolio performance responsible to determine the weight \( \lambda_{i,t} \) of model \( i \). Second, and most importantly, we performed extensive robustness checks to evaluate the gains (or lack of) from changing to past out-of-sample portfolio performance, despite higher computational cost. The results are qualitatively similar to those based on the past in-sample performance of each individual model, which makes us confident to employ this approach. The results based on an out-of-sample evaluation of individual models are available upon request.

### 3.3 Methodology for evaluating portfolio performance

We use the out-of-sample observations to evaluate the mean-variance and minimum variance portfolio performance in terms of average return (\( \hat{\mu} \)) and standard deviation (volatility) of returns (\( \hat{\sigma} \)). These statistics are calculated as follows:

\[
\hat{\mu} = \frac{1}{T - 1} \sum_{t=1}^{T-1} w_t' R_{t+1}
\]

\[
\hat{\sigma} = \sqrt{\frac{1}{T - 1} \sum_{t=1}^{T-1} (w_t' R_{t+1} - R_p)^2}.
\]

where \( R_p \) is the realized portfolio average return.

An additional aspect that must be taken into account is the impact of transaction costs on the
performance of optimal portfolios (Han, 2006). For that purpose, we follow Della Corte et al. (2008) and Thornton and Valente (2012) and compute the portfolio return net of transaction costs \( R_{p,t}^{\text{net}} \). This calculation is performed as

\[
R_{p,t}^{\text{net}} = (1 - c \cdot \text{turnover}_t) \left( 1 + w_t R_{t+1} \right) - 1,
\]

where \( c \) is the fee that must be paid for each transaction and \( \text{turnover}_t \) is the portfolio turnover at time \( t \), defined as the fraction of wealth traded between periods \( t \) and \( t + 1 \), i.e

\[
\text{turnover}_t = \sum_{j=1}^{N} (|w_{j,t+1} - w_{j,t}|).
\]

To compute (14), it is necessary to set an appropriate value for the transaction cost \( c \). French (2008) estimates that the trading cost in 2006, including “total commissions, bid-ask spreads, and other costs investors pay for trading services”, and he finds that these costs have dropped significantly over time: “from 146 basis points in 1980 to a tiny 11 basis points in 2006.” His estimate is based on stocks traded on NYSE, AMEX, and NASDAQ. We, on the other hand, consider a more conservative estimate of the transaction cost and set \( c = 15 \) basis points. We believe that setting \( c = 15 \) basis points is a conservative description of the transaction cost paid by the average investor, and professional trading firms should be able to achieve lower transaction costs.

Upon computing the average and the standard deviation of the portfolio return net of transaction costs for based on a transaction cost of 15 basis points, we report the average portfolio turnover over all out-of-sample observations as well as the risk-adjusted portfolio return net of transaction costs measured by the SR, which is defined as

\[
SR = \frac{\bar{R}_p^{\text{net}}}{\sigma^{\text{net}}},
\]

where \( \bar{R}_p^{\text{net}} \) and \( \sigma^{\text{net}} \) are, respectively, the average and the standard deviation of portfolio returns net of transaction costs.

In order to assess the relative performance of the model combination approach proposed in the paper, we consider as the main benchmark specification the equally-weighted model combination. The stationary bootstrap of Politis and Romano (1994) with \( B=1000 \) resamples and block size \( b = 5 \) was used to test the statistical significance of differences between the Sharpe Ratios and standard deviations of both mean-
variance and minimum variance portfolio returns obtained with each individual model, and with each model combination with respect to those obtained with the benchmark. The methodology suggested in Ledoit and Wolf (2008) was used to obtain p-values.

3.4 Results

Table 1 reports the performance statistics discussed in Section 3.3 for the optimal portfolios obtained with the individual models as well as with the forecast combinations. Panel A of Table 1 reports the results for the mean-variance portfolios, whereas Panel B reports the results for the minimum variance portfolios. In each panel, we report the performance of each individual candidate model and the performance of alternative forecast combination approaches. The forecast combinations are based on i) the benchmark equally-weighted combination, ii) the mean-var($\delta = 1, \eta = 1$) and min-var($\delta = 1, \eta = 1$) combinations, iii) the mean-var($\delta = 0.9, \eta = 2$) and min-var($\delta = 0.9, \eta = 2$) combinations, and iv) the method discussed in Amendola and Storti (2015) for the following loss functions: Euclidean ($L_E$), Frobenius ($L_F$), Stein ($L_S$) and asymmetric ($L_{Asym}$). Asterisks denote the instances in which the portfolio standard deviation or the portfolio SR obtained with combined forecasts are statistically different at the 5% level with respect to the one obtained with the equally-weighted model combination. The test is based on the stationary bootstrap of Politis and Romano (1994) with $B=1000$ resamples and block size $b = 5$. We assume that optimal portfolios are re-balanced on a daily basis and that the level of proportional transaction cost is $c = 15$ basis points.

Let us initially consider the performance of the mean-variance portfolios reported in Panel A of Table 1. First, when looking at the performance of individual models, the results indicate that the best performance in terms of SR is obtained with the O-GARCH and ORE specifications (0.11 and 0.09, respectively). These figures are statistically different with respect to the SR obtained with the benchmark equally-weighted forecasts combination approach (0.05). All remaining individual models underperformed the benchmark in terms of SR. The O-GARCH and ORE specifications also delivered mean-variance portfolios with lower risk and lower turnover with respect to the remaining individual specifications. These results suggest that tightly parameterized models tend to perform better when evaluated in terms of economic criteria, which corroborates the findings in Engle and Sheppard (2008).

Second, when looking at the performance of the alternative forecasting combinations approaches in Panel A, we observe that the mean-var($\delta = 1, \eta = 1$) combination delivered mean-variance portfolios with
a slightly better performance to those obtained via the equally-weighted combination. Average return, portfolio risk, turnover and SR are all very much alike. However, the most striking result in Panel A of Table 1 is that the mean-var(δ = 0.9, η = 2) combination delivered a much higher SR with respect to the benchmark (0.10 vs. 0.05). This figure is also substantially higher with respect to the majority of individual models and similar to those obtained with the best performing individual models (O-GARCH and ORE). We also observe that the mean-var(δ = 0.9, η = 2) combination delivered mean-variance portfolios with the highest average returns among all other specifications. When comparing the portfolio turnover, we observe that the mean-var(δ = 0.9, η = 2) combination leads to mean-variance portfolios with higher turnover in comparison to the equally-weighted and to the mean-var(δ = 1, η = 1) combinations, which is in line with the idea that increasing η increases transaction costs. However, the performance of the mean-var(δ = 0.9, η = 2) combination in terms of average returns more than compensates the increase in transaction costs due to the higher turnover. Finally, we also find that mean-var(δ = 0.9, η = 2) combination outperformed all other forecast combinations that rely on high-frequency data specially in terms of average and risk-adjusted returns. In particular, we observe that the best result in terms of SR obtained with the forecast combinations that rely on high-frequency data is obtained when the $L_{Asym}$ loss function is used (0.05), which is similar to the one obtained with the benchmark combination.

Let us now consider the performance of the minimum variance portfolios reported in Panel B of Table 1. In general, the results of the minimum variance portfolios resemble in many ways those reported in Panel A. First, we find that the best performing individual models in terms of portfolio risk (measured by the standard deviation) are the ORE and O-GARCH specifications (0.45%). These two specification also outperformed all other individual specifications in terms of SR (0.10 and 0.11, respectively). Second, when looking at the results of the alternative forecast combinations, we observe that the min-var(δ = 1, η = 1) specification outperformed all individual models in terms of portfolio risk. On top of that, we find that the min-var(δ = 0.9, η = 2) combination delivered minimum variance portfolios with the lowest level of risk among all other specifications (0.42%). In terms of SR, the min-var(δ = 0.9, η = 2) combination performed slightly better with respect to the benchmark (0.06 vs. 0.05). Finally, we observe that the minimum variance portfolios obtained with the combinations based on the $L_E$, $L_F$, and $L_S$ loss functions underperformed the benchmark in terms of portfolio risk, whereas the results based on the $L_{Asym}$ is similar to those obtained with the benchmark.

Taken together, the results reported in Table 1 suggest that combining alternative conditional covari-
ance forecasts bring an improvement in the performance of optimal portfolios. We find that the proposed forecast combinations delivered mean-variance portfolios with higher SR as well as minimum variance portfolios with lower risk. The differences with respect to the benchmark are statistically significant. The proposed forecast combinations outperform existing combination schemes, including not only the equally-weighted combination as well as the combinations based on existing methods that require the minimization of a loss functions and a proxy for the latent covariance matrix. This result is more evident when the proposed combination is calibrated in order to i) give more importance to recent observation (by decreasing the forgetting factor $\delta$) and ii) give more importance to the best performing individual models (by increasing the parameter $\eta$). We conclude that the use of economic criteria to devise a forecast combination scheme seems to be more appropriate when the ultimate goal is to use the resulting conditional covariance matrix to solve an economic problem.

3.4.1 Sensitivity with respect to $\delta$ and $\eta$

The previous analysis suggests that the parameters $\delta$ and $\eta$ play an important role in the performance of the proposed mean-var$(\delta, \eta)$ and min-var$(\delta, \eta)$ forecast combinations. It is still unclear, however, how robust are the results reported in Table 1 with respect to the choice of these parameters. In this sense, we conduct an additional analysis on the performance of the mean-variance and minimum variance portfolios when alternative values for both $\delta$ and $\eta$ are considered. In particular, we consider all possible combinations in which $\delta = \{1, 0.95, 0.90, 0.85\}$ and $\eta = \{1, 2, 5, 10\}$. The results for the specifications $(\delta = 1, \eta = 1)$ and $(\delta = 0.90, \eta = 2)$ are not reported since they are already contained in Table 1.

The results regarding the sensitivity analysis with respect to the choice of alternative values of $\delta$ and $\eta$ are reported in Table 2. Panel A of Table 2 reports the results for the mean-variance portfolios, whereas Panel B reports the results for the minimum variance portfolios. We find that varying the $\delta$ and $\eta$ parameters bring improvements in many instances since alternative forecast combinations outperform the simple model averaging. The results for the mean-variance portfolios reported in Panel A suggest that lowering the value of $\delta$ (i.e. putting more weight on the recent performance of individual models) and increasing the value of $\eta$ (i.e. putting more weight on the best single models) lead to improvement in terms of average and risk-adjusted returns measured by the SR. The best performance according to SR is achieved with the mean-var$(\delta = 0.85, \eta = 10)$ combination (0.13). However, one consequence of lowering the value of $\delta$ and increasing the value of $\eta$ is marked increase in portfolio turnover. For instance, the
mean-var(δ = 0.85, η = 10) combination delivered mean-variance portfolios with turnover almost 4 times higher with respect to the one obtained with the equally-weighted combination (0.58 vs. 0.15).

When looking at the results for the minimum variance portfolios reported in Panel B of Table 2, we observe that the resulting portfolio risk decreases when the value of δ decreases and the value of η increases. For instance, the lowest portfolio risk is obtained with the combination mean-var(δ = 0.85, η = 10) (0.37%). Similar as in Panel A, we see that of lowering the value of δ and increasing the value of η also increase portfolio turnover. One difference, however, among the results reported in Panels A and B is that in the latter case the the SR of the minimum variance portfolios tend to decrease for values of δ lower 0.95 and η higher than 5.

To further illustrate the impact associated with the choice of alternative values of η, we plot in Figure 3 the boxplot of the forecast combination weights when δ = 0.90 and η = {0, 2, 5, 10}. As expected, an increase in the value of η (which controls the shrinkage towards best performing models) is associated to an increase in the variability of the combination weights. Since each covariance matrix forecast imply a different portfolio allocation, frequent model changes generate large transaction costs, as seen in Table 2. Moreover, we also observe that, as expected, an increase in the value of η leads to an increase in the allocation in the best single models.

3.4.2 Alternative data sets

We now consider the performance of the mean-var(δ, η) and min-var(δ, η) forecast combinations when applied to two alternative data sets. The first alternative data set consists on daily closing prices of 45 stocks belonging to the European stock market index Eurostoxx whereas the second alternative data set consists of daily closing prices of 17 stocks belonging to the Asian stock market index STI. These two data sets have, respectively, T = 2504 and T = 2520 observations. Since intraday data for these markets are not available to us, we cannot consider the approach of Amendola and Storti (2015) as a benchmark, evidencing one of the difficulties of a more general application of this method. Implementation details discussed in Section 3.2 also applies. For the ease of notation, we refer to these two data sets as 45EURO and 17STI, respectively.

We report in Table 3 the performance of the mean-variance and minimum variance portfolios obtained with individual models and with the mean-var(δ, η) and min-var(δ, η) combinations when applied to the 45EURO data set. The performance of the mean-variance portfolios reported in Panel A suggest that the
The best individual models in terms of SR are the ORE and EWMA specifications (0.05 and 0.03, respectively). As for the forecast combinations, we see that mean-var(\(\delta = 0.90, \eta = 2\)) outperformed the equally-weighted and the mean-var(\(\delta = 1, \eta = 1\)) in terms of SR. In particular, while the benchmark combination delivered a SR roughly close to zero, the mean-var(\(\delta = 0.90, \eta = 2\)) combination achieved a SR of 0.05, which is also higher in comparison to all individual models. The performance of the minimum variance portfolios reported in Panel B of Table 3 also reveal the superiority of the proposed forecast combination. The min-var(\(\delta = 0.90, \eta = 2\)) combination delivered minimum variance portfolio with standard deviation of 0.72 while the same figures for the benchmark specifications and for the best performing individual model are, respectively, 0.76 and 0.81.

Table 4 reports the performance of the mean-variance and minimum variance portfolios obtained with individual models and with the mean-var(\(\delta, \eta\)) and min-var(\(\delta, \eta\)) combinations when applied to the 17STI data set. The results are similar to those reported in Tables 1 and 3. We find that i) the mean-variance portfolios obtained with the mean-var(\(\delta = 0.90, \eta = 2\)) combination outperformed those obtained with the benchmark combination and with the best individual model in terms of SR and ii) the minimum variance portfolios obtained with the min-var(\(\delta = 0.90, \eta = 2\)) combination also outperformed those obtained with the benchmark combination and with the best individual model in terms of portfolio standard deviation.

Finally, we report in Tables 5 and 6 a sensitivity analysis with respect to the \(\delta\) and \(\eta\) coefficients in the case of the 45EURO and 17STI data sets, respectively. The results in Panel A of Tables 5 and 6 reveal that the \(\delta\) and \(\eta\) play an important role in explaining the performance of the mean-variance portfolios. We see that lowering the value of \(\delta\) and increasing the value of \(\eta\) improve the risk-adjusted performance measured by the SR. In the case of the 45EURO data set, the best result in terms of SR is obtained with the mean-var(\(\delta = 0.85, \eta = 2\)) combination (0.06). As for the 17STI data set the best result in terms of SR is obtained with the mean-var(\(\delta = 0.85, \eta = 10\)) combination (0.08). It is worth noting that these results are substantially and statistically higher with respect to those obtained with the benchmark combination. The performance of the minimum variance portfolio reported in Panel B of Tables 5 and 6 are in the same line and points to the importance of the \(\delta\) and \(\eta\) in explaining the performance in terms of portfolio risk measured by the standard deviation. In particular, we observe that for both data sets the best minimum variance portfolio performance is obtained with the min-var(\(\delta = 0.85, \eta = 10\)) combination (0.64% and 0.58% for the 45EURO and 17STI data sets, respectively).

All in all, the additional robustness checks reported in Tables 3 to 6 are reassuring. We observe that
the proposed mean-var(δ, η) and min-var(δ, η) forecast combination are able to outperform not only the equally-weighted combination but also the majority of individual models in terms of alternative economic criteria such as portfolio risk-adjusted performance and risk when applied to two different data sets from the European and Asian stock markets. These results let us confident that the proposed approach is flexible and can yield improved results when applied to real market data.

4 Concluding remarks

Modeling and forecasting the covariance matrix of portfolio returns is of paramount importance in many economic and financial problems such as asset pricing, portfolio optimization and market risk management. In practice, however, the investor faces uncertainty on which is the most appropriate specification to model and to perform these tasks. To alleviate this problem, we put forward a novel approach to combine multivariate volatility predictions from alternative conditional covariance models. The proposed combination rule is explicitly built to exploit model complementarities and is motivated by the fact this class of models is often applied in portfolio selection problems. Four major advantages of this approach are that i) it does not require a proxy for the latent conditional covariance matrix, ii) it does not require optimization of the combination weights, iii) it holds the equally-weighted model combination as a particular case, and iv) it accommodates alternative portfolio policies as well as alternative portfolio characteristics for defining the mixing. Our empirical evidence based on three alternative data sets from different markets confirms that mean-variance and minimum variance portfolios obtained with the proposed forecast combinations have improved performance in terms of higher risk-adjusted returns and lower portfolio risk with respect to mean-variance and minimum variance portfolios obtained with a number of benchmark specifications. The results are robust to the presence of transaction costs.
An alternative version of the mean-variance optimization problem in (2) consists in adding a restriction on short sales, i.e. \( w_t \geq 0 \); see Jagannathan and Ma (2003). As a robustness check, we also implemented this alternative version of the mean-variance problem. The results are in the same line to those reported here and are available upon request. One drawback, however, of this alternative version is that a closed form solution to the optimal portfolio weights is no longer available and the problem is solved numerically, which increases computational cost.

Alternative algorithms to combine predictions for the conditional mean based on the past performance of the candidate models are proposed in Yang (2004) and Hibon and Evgeniou (2005). We detail in Section 3.2 how the past performance of each model is computed in our empirical implementation.

See Patton (2011) for a discussion on problems of using an imperfect volatility proxies for volatility forecast evaluation.

As a robustness check, we have also computed the daily realized covariance in (11) without considering the effects of overnight volatility. The results are very similar to those reported here and are available upon request.

Following the same argument as in Becker et al. (2014), it is accepted that this may not be the optimal frequency, or even method. Instead, a realized multivariate kernel could be used. It is not our objective, however, to compare among alternative realized covariance measures. Nevertheless, the realized covariance measure employed here is often found to be more accurate than the daily realized measure based only on close-to-close prices and is also very tractable. We have also assessed the robustness of our results with respect to the sampling frequency by choosing \( h = 5 \) minute and \( h = 10 \) minute. The results are very similar to those reported here and are available from the authors upon request.

One can argue that the scalar VECH in (13) and the EWMA specification share similar representations. However, our empirical results in Section 3 reveals that the resulting mean-variance and minimum variance portfolios have different characteristics, notably the turnover and the risk adjusted returns net of transaction costs measured by the SR, as it can be seen in Table 1.

We have performed robustness checks with respect to the choice of the initial estimation window. In particular, we also considered the cases in which \( t = 1000 \) and \( t = 2000 \). The results are similar to those reported here and are available upon request. We believe that choosing an initial estimation window of \( t = 1500 \) is appropriate considering that we are estimating multivariate GARCH specifications with a cross section dimension of 50 assets.

We performed extensive robustness checks regarding the choice of the block size. More specifically, we performed the test with the block size varying from 1 to 100. The results are very similar to those reported here. Moreover, we also performed the test of the statistical significance of differences in the variance of minimum variance portfolio returns. The results are also similar to those obtained with the standard deviation and are available upon request.
Figure 1: The effect of lowering $\delta$

The Figure illustrates the importance given to the most recent observations, measured by the discount factor, $\delta^t$, for alternative values for the parameter $\delta$. The Figure plots the values of $\delta^t$ for a sample with $t = 1,\ldots,100$ observations, where the $t = 1$ is the most recent observation. The Figure plots the resulting discount factor when $\delta = \{1, 0.95, 0.90, 0.85\}$. As expected, we observe an exponential decay in the importance given to the most recent observations when the value of $\delta$ decreases.

Figure 2: The effect of increasing $\eta$

The Figure illustrates the effects of increasing the value of the cross section allocation parameter $\eta$. The Figure plots the results of a small simulated experiment that works as follows. Say there are $m = 1,\ldots,10$ alternative individual conditional covariance models that delivered mean-variance portfolios according to eq. (2) with SR ranging from 0.1 to 1.0, with increments of 0.1, such that $SR_{m1} = 0.1,\ldots,SR_{m10} = 1.0$. We then compute the resulting combination weights in each of the $m$ alternative models using (5) for $\eta = \{0, 2, 4, 6, 8, 10\}$. As expected, the Figure shows that when $\eta = 0$, all models receive the same weight. When the value of $\eta$ increases, the weights on the models with higher SR increase exponentially.
The Figure illustrates the impact associated to the choice of alternative values of \( \delta \) and \( \eta \) by plotting the forecast combination weights when \( \delta = 0.90 \) and \( \eta = \{0, 2, 5, 10\} \). We observe that an increase in the value of \( \eta \) (which controls for the dispersion in cross section allocation across individual forecasts) is associated to an increase in the variability of the combination weights. Moreover, we also observe that, as expected, an increase in the value of \( \eta \) leads to an increase in the allocation in the best single models.
Table 1: Mean-variance and minimum variance portfolio performance: individual models vs. forecast combinations

The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with a set of individual models as well as with forecast combinations. The individual models are the following multivariate GARCH specifications: exponentially weighted moving average (EWMA), optimal rolling estimator (ORE), VECH, orthogonal GARCH (O-GARCH), constant conditional correlation (CCC), dynamic conditional correlation (DCC), dynamic equicorrelation (DECO), and asymmetric DCC (ASYDCC); see Section 2.3. The forecast combinations are based on i) the benchmark equally-weighted combination, ii) the mean-var(\(\delta = 1, \eta = 1\)) and min-var(\(\delta = 1, \eta = 1\)) combinations, iii) the mean-var(\(\delta = 0.9, \eta = 2\)) and min-var(\(\delta = 0.9, \eta = 2\)) combinations, and iv) the method discussed in Section 2.2 for the following loss functions: Euclidean (\(L_E\)), Frobenius (\(L_F\)), Stein (\(L_S\)) and asymmetric (\(L_{Asym}\)). SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points (b.p.). The data set consists of daily closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

<table>
<thead>
<tr>
<th>Mean return (%)</th>
<th>Std. dev. (%)</th>
<th>Turnover</th>
<th>SR (c=15 b.p.)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: mean-variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Individual models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EWMA</td>
<td>0.048</td>
<td>0.468*</td>
<td>0.111</td>
</tr>
<tr>
<td>ORE</td>
<td>0.051</td>
<td>0.455*</td>
<td>0.063</td>
</tr>
<tr>
<td>VECH</td>
<td>0.045</td>
<td>0.524*</td>
<td>0.299</td>
</tr>
<tr>
<td>O-GARCH</td>
<td>0.058</td>
<td>0.451</td>
<td>0.036</td>
</tr>
<tr>
<td>CCC</td>
<td>0.043</td>
<td>0.483*</td>
<td>0.291</td>
</tr>
<tr>
<td>DCC</td>
<td>0.048</td>
<td>0.854*</td>
<td>1.096</td>
</tr>
<tr>
<td>DECO</td>
<td>0.039</td>
<td>0.501*</td>
<td>0.181</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.046</td>
<td>0.479*</td>
<td>0.223</td>
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<tr>
<td>Combinations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.047</td>
<td>0.441</td>
<td>0.152</td>
</tr>
<tr>
<td>mean-var((\delta = 1, \eta = 1))</td>
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<td>0.441</td>
<td>0.153</td>
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<tr>
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<td>0.288</td>
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<tr>
<td>(L_E)</td>
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<td>0.353</td>
</tr>
<tr>
<td>(L_F)</td>
<td>0.048</td>
<td>0.436*</td>
<td>0.357</td>
</tr>
<tr>
<td>(L_S)</td>
<td>0.051</td>
<td>0.432*</td>
<td>0.328</td>
</tr>
<tr>
<td>(L_{Asym})</td>
<td>0.047</td>
<td>0.442</td>
<td>0.152</td>
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</table>

**Panel B: minimum variance portfolios**

<table>
<thead>
<tr>
<th>Individual models</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>EWMA</td>
<td>0.049</td>
<td>0.467*</td>
<td>0.111</td>
</tr>
<tr>
<td>ORE</td>
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<td>0.453*</td>
<td>0.062</td>
</tr>
<tr>
<td>VECH</td>
<td>0.044</td>
<td>0.524*</td>
<td>0.297</td>
</tr>
<tr>
<td>O-GARCH</td>
<td>0.056</td>
<td>0.457*</td>
<td>0.033</td>
</tr>
<tr>
<td>CCC</td>
<td>0.042</td>
<td>0.481*</td>
<td>0.284</td>
</tr>
<tr>
<td>DCC</td>
<td>0.047</td>
<td>0.876*</td>
<td>1.128</td>
</tr>
<tr>
<td>DECO</td>
<td>0.037</td>
<td>0.501*</td>
<td>0.181</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.046</td>
<td>0.476*</td>
<td>0.221</td>
</tr>
<tr>
<td>Combinations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.046</td>
<td>0.441</td>
<td>0.151</td>
</tr>
<tr>
<td>min-var((\delta = 1, \eta = 1))</td>
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<td>0.437*</td>
<td>0.132</td>
</tr>
<tr>
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</tr>
<tr>
<td>(L_E)</td>
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<td>0.358</td>
</tr>
<tr>
<td>(L_F)</td>
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<td>0.362</td>
</tr>
<tr>
<td>(L_S)</td>
<td>0.049</td>
<td>0.433*</td>
<td>0.330</td>
</tr>
<tr>
<td>(L_{Asym})</td>
<td>0.046</td>
<td>0.441</td>
<td>0.151</td>
</tr>
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</table>
Table 2: Performance of mean-var(\(\delta, \eta\)) and min-var(\(\delta, \eta\)) forecast combinations under alternative values of \(\delta\) and \(\eta\)

The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with the forecast combinations based on the mean-var(\(\delta, \eta\)) and min-var(\(\delta, \eta\)) combination rules in (6) and (9) respectively, for \(\delta = \{1, 0.95, 0.90, 0.85\}\) and \(\eta = \{1, 2, 5, 10\}\). SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points. The data set consists of daily closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

<table>
<thead>
<tr>
<th></th>
<th>Mean return (%)</th>
<th>Std. dev. (%)</th>
<th>Turnover</th>
<th>SR (c=15 b.p.)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: mean-variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean-var((\delta=1, \eta=2))</td>
<td>0.049</td>
<td>0.442</td>
<td>0.154</td>
<td>0.059</td>
</tr>
<tr>
<td>mean-var((\delta=1, \eta=5))</td>
<td>0.049</td>
<td>0.447</td>
<td>0.160</td>
<td>0.056</td>
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<tr>
<td>mean-var((\delta=1, \eta=10))</td>
<td>0.047</td>
<td>0.457*</td>
<td>0.166</td>
<td>0.049</td>
</tr>
<tr>
<td>mean-var((\delta=0.95, \eta=1))</td>
<td>0.063</td>
<td>0.443</td>
<td>0.188</td>
<td>0.078*</td>
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<tr>
<td>mean-var((\delta=0.95, \eta=2))</td>
<td>0.074</td>
<td>0.449*</td>
<td>0.236</td>
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<td>mean-var((\delta=0.95, \eta=5))</td>
<td>0.095</td>
<td>0.466*</td>
<td>0.333</td>
<td>0.096*</td>
</tr>
<tr>
<td>mean-var((\delta=0.95, \eta=10))</td>
<td>0.109</td>
<td>0.484*</td>
<td>0.406</td>
<td>0.099*</td>
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<td>mean-var((\delta=0.90, \eta=5))</td>
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<td>0.465*</td>
<td>0.419</td>
<td>0.109*</td>
</tr>
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<td>mean-var((\delta=0.90, \eta=10))</td>
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<td>0.483*</td>
<td>0.518</td>
<td>0.111*</td>
</tr>
<tr>
<td>mean-var((\delta=0.85, \eta=1))</td>
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<td>0.446</td>
<td>0.339</td>
<td>0.105*</td>
</tr>
<tr>
<td>mean-var((\delta=0.85, \eta=5))</td>
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<td>0.461*</td>
<td>0.499</td>
<td>0.123*</td>
</tr>
<tr>
<td>mean-var((\delta=0.85, \eta=10))</td>
<td>0.148</td>
<td>0.476*</td>
<td>0.578</td>
<td>0.128*</td>
</tr>
<tr>
<td><strong>Panel B: minimum variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>min-var((\delta=1, \eta=2))</td>
<td>0.047</td>
<td>0.436*</td>
<td>0.127</td>
<td>0.065*</td>
</tr>
<tr>
<td>min-var((\delta=1, \eta=5))</td>
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<td>0.438</td>
<td>0.139</td>
<td>0.060</td>
</tr>
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<td>min-var((\delta=1, \eta=10))</td>
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<td>0.447</td>
<td>0.180</td>
<td>0.041</td>
</tr>
<tr>
<td>min-var((\delta=0.95, \eta=1))</td>
<td>0.047</td>
<td>0.432*</td>
<td>0.125</td>
<td>0.065*</td>
</tr>
<tr>
<td>min-var((\delta=0.95, \eta=2))</td>
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<td>0.426*</td>
<td>0.116</td>
<td>0.070*</td>
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<td>min-var((\delta=0.95, \eta=5))</td>
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<td>0.415*</td>
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<td>0.070*</td>
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<td>0.407*</td>
<td>0.136</td>
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<td>0.428*</td>
<td>0.129</td>
<td>0.063*</td>
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<td>0.402*</td>
<td>0.155</td>
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<td>0.035</td>
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<td>0.413*</td>
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<td>min-var((\delta=0.85, \eta=10))</td>
<td>0.044</td>
<td>0.376*</td>
<td>0.255</td>
<td>0.015*</td>
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</table>
The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with a set of individual models as well as with forecast combinations. The individual models are the following multivariate GARCH specifications: exponentially weighted moving average (EWMA), optimal rolling estimator (ORE), VECH, orthogonal GARCH (O-GARCH), constant conditional correlation (CCC), dynamic conditional correlation (DCC), dynamic equicorrelation (DECO), and asymmetric DCC (ASYDCC); see Section 2.3. The forecast combinations are based on i) the benchmark equally-weighted combination, ii) the mean-var($\delta = 1$, $\eta = 1$) and min-var($\delta = 1$, $\eta = 1$) combinations, and iii) the mean-var($\delta = 0.9$, $\eta = 2$) and min-var($\delta = 0.9$, $\eta = 2$) combinations. SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points (b.p.). The data set consists of daily closing prices of 45 stocks belonging to the European stock market index Eurostoxx from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

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<th>Std. dev. (%)</th>
<th>Turnover</th>
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<tbody>
<tr>
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<td></td>
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<tr>
<td><strong>Individual models</strong></td>
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<td>0.954*</td>
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<tr>
<td>CCC</td>
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<td>-0.262*</td>
</tr>
<tr>
<td>DECO</td>
<td>0.038</td>
<td>0.865*</td>
<td>0.416</td>
<td>-0.026</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.059</td>
<td>0.841*</td>
<td>0.560</td>
<td>-0.028*</td>
</tr>
<tr>
<td><strong>Combinations</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.055</td>
<td>0.754</td>
<td>0.346</td>
<td>0.006</td>
</tr>
<tr>
<td>mean-var($\delta=1,\eta=1$)</td>
<td>0.057</td>
<td>0.752</td>
<td>0.309</td>
<td>0.016*</td>
</tr>
<tr>
<td>mean-var($\delta=0.9,\eta=2$)</td>
<td>0.158</td>
<td>0.776*</td>
<td>0.782</td>
<td>0.054*</td>
</tr>
<tr>
<td><strong>Panel B: minimum variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Individual models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EWMA</td>
<td>0.060</td>
<td>0.814*</td>
<td>0.212</td>
<td>0.036*</td>
</tr>
<tr>
<td>ORE</td>
<td>0.057</td>
<td>0.810*</td>
<td>0.100</td>
<td>0.053*</td>
</tr>
<tr>
<td>VECH</td>
<td>0.054</td>
<td>0.843*</td>
<td>0.549</td>
<td>-0.033*</td>
</tr>
<tr>
<td>O-GARCH</td>
<td>0.032</td>
<td>0.952*</td>
<td>0.101</td>
<td>0.019</td>
</tr>
<tr>
<td>CCC</td>
<td>0.060</td>
<td>0.815*</td>
<td>0.478</td>
<td>-0.013</td>
</tr>
<tr>
<td>DCC</td>
<td>0.004</td>
<td>1.460*</td>
<td>2.734</td>
<td>-0.271*</td>
</tr>
<tr>
<td>DECO</td>
<td>0.044</td>
<td>0.909*</td>
<td>0.399</td>
<td>-0.016</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.056</td>
<td>0.839*</td>
<td>0.556</td>
<td>-0.031*</td>
</tr>
<tr>
<td><strong>Combinations</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.054</td>
<td>0.759</td>
<td>0.346</td>
<td>0.005</td>
</tr>
<tr>
<td>min-var($\delta=1,\eta=1$)</td>
<td>0.056</td>
<td>0.757</td>
<td>0.325</td>
<td>0.012*</td>
</tr>
<tr>
<td>min-var($\delta=0.9,\eta=2$)</td>
<td>0.058</td>
<td>0.724*</td>
<td>0.329</td>
<td>0.014*</td>
</tr>
</tbody>
</table>
The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with a set of individual models as well as with forecast combinations. The individual models are the following multivariate GARCH specifications: exponentially weighted moving average (EWMA), optimal rolling estimator (ORE), VECH, orthogonal GARCH (O-GARCH), constant conditional correlation (CCC), dynamic conditional correlation (DCC), dynamic equicorrelation (DECO), and asymmetric DCC (ASYDCC); see Section 2.3. The forecast combinations are based on i) the benchmark equally-weighted combination, ii) the mean-var($\delta = 1, \eta = 1$) and min-var($\delta = 1, \eta = 1$) combinations, and iii) the mean-var($\delta = 0.9, \eta = 2$) and min-var($\delta = 0.9, \eta = 2$) combinations. SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points (b.p.). The data set consists of daily closing prices of 17 stocks belonging to the Asian stock market index STI from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

<table>
<thead>
<tr>
<th>Mean return (%)</th>
<th>Std. dev. (%)</th>
<th>Turnover</th>
<th>SR (c=15 b.p.)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: mean-variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Individual models</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EWMA</td>
<td>0.033</td>
<td>0.727*</td>
<td>0.061</td>
</tr>
<tr>
<td>ORE</td>
<td>0.033</td>
<td>0.729*</td>
<td>0.057</td>
</tr>
<tr>
<td>VECH</td>
<td>0.035</td>
<td>0.734*</td>
<td>0.213</td>
</tr>
<tr>
<td>O-GARCH</td>
<td>0.031</td>
<td>0.756*</td>
<td>0.053</td>
</tr>
<tr>
<td>CCC</td>
<td>0.045</td>
<td>0.710</td>
<td>0.197</td>
</tr>
<tr>
<td>DCC</td>
<td>0.035</td>
<td>1.154*</td>
<td>1.246</td>
</tr>
<tr>
<td>DECO</td>
<td>0.031</td>
<td>0.724*</td>
<td>0.176</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.041</td>
<td>0.715*</td>
<td>0.145</td>
</tr>
<tr>
<td><strong>Combinations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.038</td>
<td>0.698</td>
<td>0.147</td>
</tr>
<tr>
<td>mean-var($\delta=1,\eta=1$)</td>
<td>0.038</td>
<td>0.696</td>
<td>0.124</td>
</tr>
<tr>
<td>mean-var($\delta=0.90,\eta=2$)</td>
<td>0.086</td>
<td>0.716*</td>
<td>0.324</td>
</tr>
<tr>
<td><strong>Panel B: minimum variance portfolios</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Individual models</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EWMA</td>
<td>0.033</td>
<td>0.705*</td>
<td>0.060</td>
</tr>
<tr>
<td>ORE</td>
<td>0.033</td>
<td>0.707*</td>
<td>0.056</td>
</tr>
<tr>
<td>VECH</td>
<td>0.033</td>
<td>0.703*</td>
<td>0.207</td>
</tr>
<tr>
<td>O-GARCH</td>
<td>0.031</td>
<td>0.727*</td>
<td>0.050</td>
</tr>
<tr>
<td>CCC</td>
<td>0.042</td>
<td>0.689*</td>
<td>0.205</td>
</tr>
<tr>
<td>DCC</td>
<td>0.042</td>
<td>1.154*</td>
<td>1.214</td>
</tr>
<tr>
<td>DECO</td>
<td>0.023</td>
<td>0.732*</td>
<td>0.171</td>
</tr>
<tr>
<td>ASY-DCC</td>
<td>0.039</td>
<td>0.699*</td>
<td>0.149</td>
</tr>
<tr>
<td><strong>Combinations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equally-weighted (benchmark)</td>
<td>0.035</td>
<td>0.671</td>
<td>0.146</td>
</tr>
<tr>
<td>min-var($\delta=1,\eta=1$)</td>
<td>0.036</td>
<td>0.670</td>
<td>0.122</td>
</tr>
<tr>
<td>min-var($\delta=0.90,\eta=2$)</td>
<td>0.032</td>
<td>0.646*</td>
<td>0.125</td>
</tr>
</tbody>
</table>
The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with the forecast combinations based on the mean-var(δ, η) and min-var(δ, η) combination rules in (6) and (9) respectively, for δ = {1, 0.95, 0.90, 0.85} and η = {1, 2, 5, 10}. SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points. The data set consists of daily closing prices of 45 stocks belonging to the European stock market index Eurostoxx from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

### Table 6: Performance of mean-var(δ, η) and min-var(δ, η) forecast combinations under alternative values of δ and η: 17STI data set

The Table reports performance statistics for mean-variance (Panel A) and minimum variance (Panel B) portfolios obtained with the forecast combinations based on the mean-var(δ, η) and min-var(δ, η) combination rules in (6) and (9) respectively, for δ = {1, 0.95, 0.90, 0.85} and η = {1, 2, 5, 10}. SR denotes the Sharpe ratio, which is computed using returns net of transaction costs of 15 basis points. The data set consists of daily closing prices of 17 stocks belonging to the Asian stock market index STI from 01/01/2004 until 31/12/2013. All figures are based on out-of-sample observations. Optimal portfolios are re-balanced on a daily basis. Asterisks indicate that the figure is statistically different than the one obtained with the equally-weighted model combination at a significance level of 5%.

<table>
<thead>
<tr>
<th>Panel A: mean-variance portfolios</th>
<th>Panel B: minimum variance portfolios</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return (%)</td>
<td>Std. dev. (%)</td>
</tr>
<tr>
<td>mean-var(δ=1, η=2)</td>
<td>0.039</td>
</tr>
<tr>
<td>mean-var(δ=1, η=5)</td>
<td>0.039</td>
</tr>
<tr>
<td>mean-var(δ=1, η=10)</td>
<td>0.040</td>
</tr>
<tr>
<td>mean-var(δ=0.95, η=1)</td>
<td>0.054</td>
</tr>
<tr>
<td>mean-var(δ=0.95, η=2)</td>
<td>0.069</td>
</tr>
<tr>
<td>mean-var(δ=0.95, η=5)</td>
<td>0.088</td>
</tr>
<tr>
<td>mean-var(δ=0.95, η=10)</td>
<td>0.101</td>
</tr>
<tr>
<td>mean-var(δ=0.90, η=1)</td>
<td>0.063</td>
</tr>
<tr>
<td>mean-var(δ=0.90, η=2)</td>
<td>0.119</td>
</tr>
<tr>
<td>mean-var(δ=0.90, η=5)</td>
<td>0.135</td>
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<tr>
<td>mean-var(δ=0.90, η=10)</td>
<td>0.072</td>
</tr>
<tr>
<td>mean-var(δ=0.85, η=2)</td>
<td>0.101</td>
</tr>
<tr>
<td>mean-var(δ=0.85, η=5)</td>
<td>0.142</td>
</tr>
<tr>
<td>mean-var(δ=0.85, η=10)</td>
<td>0.159</td>
</tr>
</tbody>
</table>
References


